Algebraicity and transcendence of power series: combinatorial and computational aspects

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informatics mathematics Inria

Algorithmic and Enumerative Combinatorics
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Overview

1. Monday: Context and Examples
2. Tuesday: Properties and Criteria (1)
3. Wednesday: Properties and Criteria (2)
4. Thursday: Algorithmic Proofs of Algebraicity
5. Friday: Transcendence in Lattice Path Combinatorics

Part II: Properties and Criteria (2)

Arithmetic properties

Algebraic series have almost integer coefficients

Theorem [Eisenstein, 1852], [Heine, 1853]
Any algebraic power series $f = \sum_{n \geq 0} a_n t^n$ in $\mathbb{Q}[[t]]$ is globally bounded:
there exists an integer $C > 0$ such that $a_n C^n$ is an integer for all $n \geq 1$.

"A premature death prevented Eisenstein from presenting the proof of this important theorem"

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▷ In particular, denominators of a_n's admit only finitely many prime divisors

▷ Proof idea: Furstenberg's theorem + "diagonals are globally bounded"

▷ $\sum t^n/n$, $\sum t^n/(n^2+1)$ and $\sum_n t^n/n!$ are transcendental

▷ Research problems:

- Christol's conj. (1990): Is any D-finite glob. bounded series a diagonal?
- Concrete subproblem: is ${}_3F_2\left(\begin{matrix}\frac{1}{9} & \frac{4}{9} & \frac{5}{9} \\ \frac{1}{3} & 1 & \end{matrix}\,\middle|\, 729\,t\right)$ a diagonal?

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▷ The smallest possible constant C is called *Eisenstein constant* of f.

▷ Best current bound [Dwork, van der Poorten 1992]

$$C \leq 4.8 \left(8\, e^{-3}\, D^{4+2.74\log D}\, e^{1.22D}\right)^{D} \cdot H^{2D-1} \quad = e^{O(D^2)} \cdot H^{2D-1}$$

where D is the degree of the (minpoly of) f, and H the max of its coeffs.

▷ Research problems:

- Is this bound (asymptotically) tight?
- Find a (fast) algorithm for computing C.

The coefficient sequence of an algebraic series is p-automatic

Theorem [Christol, 1979], [Christol, Kamae, Mendès France, Rauzy, 1980]
If $f = \sum_{n \geq 0} a_n t^n$ in $\mathbb{Z}[[t]]$ is algebraic, then for any prime number p the sequence $(a_n \bmod p)_{n \geq 0}$ is p-automatic, i.e., there exists a finite automaton with input the base-p expansion of n and output the value $a_n \bmod p$.

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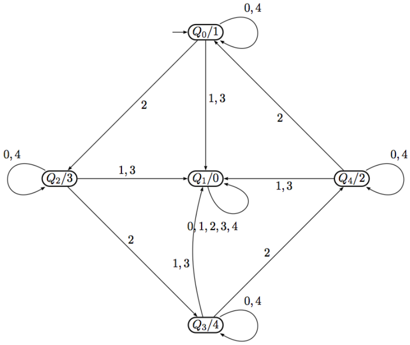
▷ Converse is false:



A 5-automaton for the Apéry sequence $a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \bmod 5$

▷ E.g., $a_{40164320} \bmod 5 = 1$, since $40164320 = (40240224240)_5$

▷ More generally, $a_n \equiv 1 \bmod 5$ iff the base-5 expansion of n does not contain the digits 1 and 3 and if the number of 2's is a multiple of 4

The coefficient sequence of an algebraic series is p-automatic

Theorem [Christol, 1979], [Christol, Kamae, Mendès France, Rauzy, 1980]
Let p be a prime number, and let $f = \sum_{n\geq 0} a_n t^n$ be a power series in $\mathbb{F}_p[[t]]$.
The following assertions are equivalent:

1. f is algebraic over $\mathbb{F}_p(t)$;
2. the coefficients sequence $(a_n)_{n\geq 0}$ is p-automatic;
3. f satisfies a Mahler equation

$$c_0(t)f(t) + c_1(t)f(t^p) + \cdots + c_r(t)f(t^{p^r}) = 0, \quad \text{for some } c_j \in \mathbb{F}_p[t], c_0 \neq 0.$$

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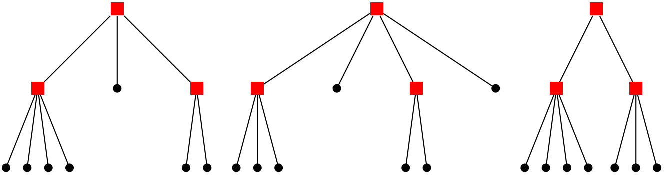
▷ Useful in transcendence in conjunction with Cobham's Theorem [1969]:
If a sequence on a finite set is both a-automatic and b-automatic, and if $\log a / \log b \notin \mathbb{Q}$, then the sequence is ultimately periodic.

▷ [Allouche, 1995] The Thue-Morse series T is algebraic over $\mathbb{F}_q(t)$ iff $q = 2^s$

▷ [Shallit, 1999] If a sequence (a_n) only takes values 4 and 7, and if $s_2(n)$ is the sum of the base-2 digits of n, then $f = \sum_n 2^{s_2(n)} a_n^n t^n$ is transcendental.

Fast computation of the Nth term modulo p

Problem: count 2-3-4 trees $\longrightarrow$ f_n = nb. of trees with n internal nodes



▷ GF: $f = \sum_n f_n t^n = 1 + 3t + 27t^2 + 333t^3 + 4752t^4 + 73764t^5 + \cdots$,
root of $P(t,T) = T - 1 - t(T^2 + T^3 + T^4)$

▷ Abel's theorem + linear recurrence $\Longrightarrow$ computation of f_N in $O(N)$ ops.

▷ Mahler equation $t + f(t) + (t^2 + t + 1)f(t^2) + tf(t^4) + t^2 f(t^8) = 0 \mod 2$

▷ $f_n \bmod 2 = \begin{cases} f_{(n-1)/2} \bmod 2, & \text{if } n \equiv 3 \bmod 4. \\ f_{(n-1)/2} + f_{(n-1)/4} \bmod 2, & \text{if } n \equiv 1 \bmod 4, \\ f_{n/2} + f_{n/2-1} + f_{(n-2)/8} \bmod 2, & \text{if } n \equiv 2 \bmod 8, \\ f_{n/2} + f_{n/2-1} \bmod 2, & \text{else.} \end{cases}$

▷ Computation of f_N modulo 2 in $O(\log N)$ operations.

Diagonals and Hadamard products modulo a prime

Just as in the midst of life there is death, so in the midst of algebraicity — at least over finite fields — there is transcendence.

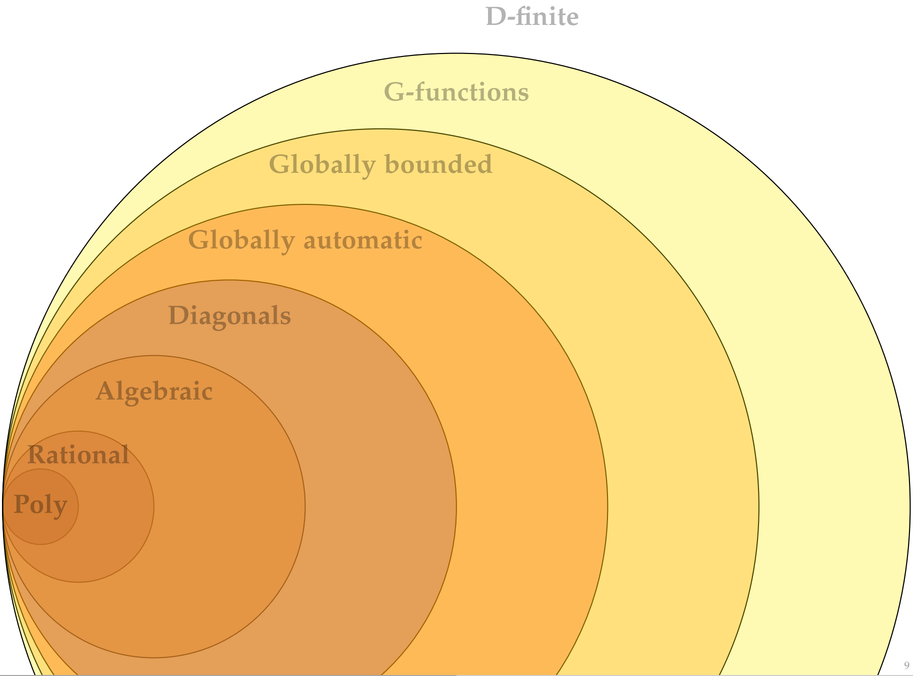
[van der Poorten, 1991]

Theorem [Furstenberg, 1967], [Deligne, 1984] Modulo a prime number p:

- The Hadamard product of algebraic series is algebraic
- The diagonal of a rational function of several variables is algebraic
- The diagonal of an algebraic function of several variables is algebraic

$$\triangleright\ \mathrm{Diag}\left(\frac{1}{1-x-y-z}\right) = \sum_{n\geq 0} \frac{(3n)!}{n!^3} t^n = (8t^3+2t^2+6t+1)^{-\frac{1}{10}} \bmod 11$$

$$\triangleright\ \frac{1}{\sqrt{1-t}} \odot \frac{1}{\sqrt{1-t}} = {}_2F_1\left(\left.\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}\right| t\right) = (t^5+3t^4+t^3+t^2+3t+1)^{-\frac{1}{10}} \bmod 11$$


D-finite
G-functions
Globally bounded
Globally automatic
Diagonals
Algebraic
Rational
Poly

Conjecture [Grothendieck, 1960's, unpublished; Katz, 1972]
Let $A \in \mathbb{Q}(t)^{r\times r}$. The following assertions are equivalent:

- The system (S) $y' = Ay$ has a full set of algebraic solutions
- For almost all prime numbers p, the system (S_p) defined by reduction of (S) modulo p has a full set of algebraic solutions over $\mathbb{F}_p(t)$
- $A_p = 0 \bmod p$ for almost all primes p, where A_p = p-curvature of (S):

$$A_0 = I_r, \quad \text{and} \quad A_{\ell+1} = A_\ell' + A_\ell A \quad \text{for} \quad \ell \geq 0.$$

▷ Proved by [Katz 1982] for *Picard-Fuchs systems*, but still widely open in the general case (even for $r = 2$)

▷ For each p, the last condition can be checked algorithmically

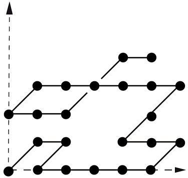
▷ [B., Caruso, Schost, 2015] Fast algorithms for the p-curvature

A combinatorial application: Gessel's conjecture

- Gessel walks: walks in $\mathbb{N}^2$ using only steps in $\mathcal{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$
- $g(i,j,n)$ = number of walks from $(0,0)$ to (i,j) with n steps in $\mathcal{S}$

Question: Nature of the generating function

$$G(x,y,t) = \sum_{i,j,n=0}^{\infty} g(i,j,n)\, x^i y^j t^n \;\in \mathbb{Q}[[x,y,t]]$$



Theorem [B. & Kauers 2010] $G(x,y,t)$ is an algebraic power series†.

→ Effective, computer-driven discovery and proof ⟶ Thursday
→ Key step in discovery: p-curvature computation of two 11th order (guessed) differential operators for $G(x,0,t)$, and $G(0,y,t)$

† Minimal polynomial $P(x,y,t,G(t;x,y)) = 0$ has $> 10^{11}$ terms; $\approx 30\,$Gb (!)

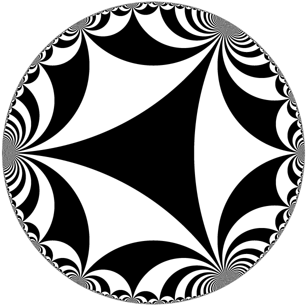
Theorem [Schwarz, 1873]

Let $a, b, c \in \mathbb{Q}$, s.t. $a, b, c-a, c-b \notin \mathbb{Z}$. Set $(\lambda, \mu, \nu) = (1-c, c-a-b, b-a)$. Up to permutations and sign changes of λ, μ, ν, and addition to (λ, μ, ν) of $(\ell, m, n) \in \mathbb{Z}^3$ with $\ell + m + n$ even, a table gives all algebraic ${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \middle| t\right)$'s.

Tabelle

enthaltend, abgesehen vom gemeinsamen Factor π, die Bogenzahlen der Winkel und den Flächeninhalt der reducirten sphärischen Dreiecke, welche auf einer Kugeloberfläche vom Radius 1 durch die Symmetrieebenen einer concentrischen regelmässigen Doppelpyramide oder eines concentrischen regelmässigen Polyeders bestimmt werden.

No.	λ''	μ''	ν''	$\frac{\text{Inhalt}}{\pi}$	Polyeder
I.	$\frac{1}{2}$	$\frac{1}{2}$	ν	ν	Regelmässige Doppelpyramide
II.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6} = A$	Tetraeder
III.	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3} = 2A$	
IV.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{12} = B$	Würfel und Oktaeder
V.	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6} = 2B$	
VI.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{30} = C$	Dodekaeder und Ikosaeder
VII.	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{15} = 2C$	
VIII.	$\frac{2}{3}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{15} = 2C$	
IX.	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{10} = 3C$	
X.	$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{2}{15} = 4C$	
XI.	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{5} = 6C$	
XII.	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{5} = 6C$	
XIII.	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5} = 6C$	
XIV.	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{7}{30} = 7C$	
XV.	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3} = 10C$	



- ▷ Proof based on geometric arguments (sphere tilings by spherical triangles)
- ▷ Basic case: ${}_2F_1\left(\begin{matrix} r & 1-r \\ & \frac{1}{2} \end{matrix} \middle| t\right) = \dfrac{\cos((1-2r) \cdot \arcsin(\sqrt{t}))}{\sqrt{1-t}}$, $r \in \mathbb{Q}$ + sporadic cases

Whatever the beauty of Schwarz's result, one must recognize that it is achieved through a long detour. [Kampé de Fériet, 1937]

Theorem [Landau, 1904], [Stridsberg, 1911], [Landau, 1911], [Errera, 1913]
Assume $a, b, c \in \mathbb{Q}$ such that $a, b, c-a, c-b \notin \mathbb{Z}$. Then ${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \,\middle|\, t\right)$ is algebraic if and only if for every r coprime with the denominators of a, b and c, either $\{ra\} \leq \{rc\} < \{rb\}$ or $\{rb\} \leq \{rc\} < \{ra\}$. $\quad (\{x\} \stackrel{\text{def}}{=} x - \lfloor x \rfloor)$

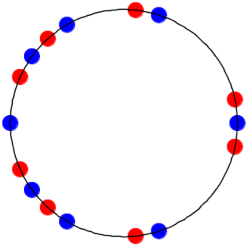
▷ Proof based on Eisenstein's theorem.

▷ $\dfrac{{}_2F_1\left(\begin{matrix} -\frac{1}{2} & -\frac{1}{6} \\ & \frac{2}{3} \end{matrix} \,\middle|\, 16\,t\right) - 1}{2t} = 1 + 2\,t + 11\,t^2 + 85\,t^3 + 782\,t^4 + \cdots$ is algebraic

▷ ${}_2F_1\left(\begin{matrix} \frac{1}{12} & \frac{5}{12} \\ & 1 \end{matrix} \,\middle|\, 1728\,t\right) = 1 + 60\,t + 39780\,t^2 + 38454000\,t^3 + \cdots$ not algebraic

Theorem [Beukers, Heckman, 1989]

Let $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_{k-1}, b_k = 1\}$ be two sets of rational parameters, assumed disjoint modulo $\mathbb{Z}$. Let D be their common denominator. Then $_kF_{k-1}\left(\begin{matrix} a_1 & a_2 & \cdots & a_k \\ b_1 & \cdots & b_{k-1} & \end{matrix} \middle| \, t\right)$ is algebraic iff $\{e^{2i\pi r a_j}, j \leq k\}$ and $\{e^{2i\pi r b_j}, j < k\}$ interlace on the unit circle for all $1 \leq r < D$ with $\gcd(r, D) = 1$.



▷ $\displaystyle\sum_n \frac{(30n)!n!}{(15n)!(10n)!(6n)!} t^n = {}_8F_7\left(\begin{matrix} \frac{1}{30} & \frac{7}{30} & \frac{11}{30} & \frac{13}{30} & \frac{17}{30} & \frac{19}{30} & \frac{23}{30} & \frac{29}{30} \\ \frac{1}{5} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{4}{5} & \end{matrix} \middle| \, 2^{14}\, 3^9\, 5^5\, t\right)$ is algebraic

Theorem [Rodriguez-Villegas, 2005]
Let $(\gamma_\kappa)_{\kappa\geq 0}$ an integer sequence with finitely many non-zero terms. Let

$$a_n = \prod_{\kappa\geq 1}(\kappa n)!^{\gamma_\kappa}, \qquad f = \sum_n a_n t^n \in \mathbb{Q}[[t]].$$

Then f is algebraic if and only if $f \in \mathbb{Z}[[t]]$ and $\sum_\kappa \kappa\gamma_\kappa = 0$ and $\sum_\kappa \gamma_\kappa = -1$.

▷ If $\gcd(a,b) = 1$, then $f = \sum_n \frac{(an+bn)!}{(an)!(bn)!} t^n$ is algebraic of degree $\binom{a+b}{a}$.

▷ $f = \sum_n \frac{(2n)!(5n)!^2}{(3n)!^4} t^n$ is transcendental.

▷ $f = \sum_n \frac{(30n)!n!}{(15n)!(10n)!(6n)!} t^n$ is algebraic of degree $483,840$ (!)

▷ Bonus [Bober, 2009]: classification of integral ratios of factorial products

A sequence $(a_n)_n$ of rational numbers is called p-Lucas (p prime number) if

- all the denominators of the a_n's are prime to p;
- $a_{pi+j} \equiv a_i a_j \bmod p$ for all $i \geq 0$ and $0 \leq j < p$.

Theorem [Allouche, Gouyou-Beauchamps, Skordev, 1998]
For $f = \sum_n a_n t^n$ in $\mathbb{Q}[[t]] \setminus \{0\}$, the following conditions are equivalent:

1. f is algebraic and (a_n) has the p-Lucas property for all large primes p;
2. There exists $P \in \mathbb{Q}[t]$ of degree at most 2, with $P(0) = 1$ and $f = \frac{1}{\sqrt{P(t)}}$;
3. Either $(a_n) = \binom{2n}{n} a^n$, or $a_n = P_n(a) b^n$ with $ab, b^2 \in \mathbb{Q}$, $P_n = \frac{1}{2^n n!} \frac{\partial^n (t^2-1)^n}{\partial t^n}$.

▷ Starting point: if $(a_n)_n$ is p-Lucas, then $f = \sum_n a_n t^n$ is algebraic over $\mathbb{F}_p(t)$:

$$f = (a_0 + \cdots + a_{p-1} t^{p-1}) \times f^p.$$

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▷ Corollary: if $a_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$, then $f = \sum_n a_n t^n$ is transcendental

▷ Corollary: if $r_1, \ldots, r_m$ are positive integers, then

$$f = \sum_{n \geq 0} \binom{2n}{n}^{r_1} \binom{3n}{n}^{r_2} \cdots \binom{(m+1)n}{n}^{r_m} t^n$$

is algebraic if and only if $m = 1$ and $r_1 = 1$.

Analytic properties

Es ist eine Tatsache, daß die genauere Kenntnis des Verhaltens einer analytischen Funktion in der Nähe ihrer singulären Stellen eine Quelle von arithmetischen Sätzen ist.

It is a fact that the exact knowledge of the behavior of an analytic function near their singular points is a source of arithmetic theorems.

[Hecke, 1923]

Gap theorems for lacunary series

Roughly: "lacunary series" = series in which groups of arbitrarily many successive zero coefficients occur infinitely often

Theorem [Hadamard, 1892], [Fabry, 1896], [Faber, 1906]
Let $(\lambda_n)_n$ be an increasing sequence of natural numbers. If $\lambda_n/n \to \infty$, then any power series $f = \sum_{n\geq 0} a_n t^{\lambda_n}$ in $\mathbb{C}[[t]]$ with finite positive convergence radius admits the convergence circle as a natural boundary.

▷ Bonus [Pólya, 1939]: the converse is also true, so the result is optimal

▷ $\sum_n t^{2^n}$ is transcendental

▷ $\sum_n t^{n!}$ is transcendental

▷ the theta series $\sum_n t^{n^2}$ is transcendental

▷ $\sum_n t^{p_n}$ is transcendental, where p_n is the nth prime number

Theorem

Assume $\sum_{n\geq 0} a_n t^{\lambda_n} \in \mathbb{Q}[[t]]$ is algebraic, where $(\lambda_n)_n$ is an increasing sequence of integers. There is a $C > 0$ such that $\lambda_{n+1} - \lambda_n \leq C$ for $n \geq 1$.

▷ Same conclusion holds for any D-finite function

▷ Arithmetic proof: consequence of rational approximation results

▷ Analytic proof: consequence of Fabry's theorem

▷ Algebraic proof: exploits linear recurrence on coefficients

▷ Research problem: find tight bounds C (e.g., [Dutter, 2015])

Bonus: puzzling gap conjectures

Rigidity Conjecture [Furter, 2015]
Let $f = t(1 + a_1 t + \cdots + a_d t^m) \in \mathbb{Q}[t]$. If m consecutive coefficients of its compositional inverse $f^{[-1]} \in \mathbb{Q}[[t]]$ vanish, then $f = t$.

▷ Only proved for $m = 1$ and $m = 2$.

Gap conjecture [B., 2015] Let $f(t)$ in $\mathbb{Q}[[t]] \setminus \mathbb{Q}(t)$ be algebraic, with minimal polynomial $P(t,T) \in \mathbb{Q}[t,T]$. Then f admits at most $\deg_t P(\deg_T P - 1) - 1$ zero consecutive coefficients. Equality for $P = T^D - T + t^d$, $f = t^d + t^{dD} + \cdots$.

Theorem [Flajolet, 1987]
If $f(t) = \sum_n a_n t^n \in \mathbb{Q}[[t]]$ is algebraic, then a_n has an asymptotic equivalent

$$a_n = \frac{\rho^n n^\alpha}{\Gamma(\alpha+1)} \cdot \sum_{i=0}^{m} C_i \omega_i^n + O(\rho^n n^\beta),$$

where $\alpha \in \mathbb{Q} \setminus \{-1,-2,-3,\dots\}$; $\beta < \alpha$; $\rho \in \overline{\mathbb{Q}}_{>0}$; $C_i, \omega_i \in \overline{\mathbb{Q}}$ and $|\omega_i| = 1$

Proof ingredients: Newton-Puiseux; transfer based on Cauchy's formula (from local behaviour at singularities to asymptotics of coefficients); and

$$[t^n](1-t)^d = \binom{n+d-1}{d-1} \sim \frac{n^{d-1}}{\Gamma(d)} \quad \text{(Stirling)}$$

Corollary If $a_n \sim \gamma \rho^n n^\alpha$ and either
(i) $\alpha \in \mathbb{Z}_{<0}$; (ii) $\alpha \notin \mathbb{Q}$; (iii) $\rho \notin \overline{\mathbb{Q}}$; (iv) $\gamma \cdot \Gamma(\alpha+1) \notin \overline{\mathbb{Q}}$
then f is transcendental.

▷ $\sum_n a_n t^n = \mathrm{Diag}\left(\frac{1}{1-x-y-z}\right)$ is transcendental: $a_n = \frac{(3n)!}{n!^3} \sim 3^{3n}\frac{\sqrt{3}}{2\pi n}$

▷ GF of partitions $\sum_{n=0}^{\infty} p(n)t^n$ is transcendental: $p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$

▷ $f = \sum_n p_n t^n$ is transcendental by the prime number theorem $p_n \sim n \log n$.

▷ The Apéry series $\sum a_n t^n$ with $a_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$ is transcendental, since

$$a_n \sim \frac{(1+\sqrt{2})^{4n+2}}{2^{9/4}\pi^{3/2}n^{3/2}}, \quad \text{and} \quad \frac{\Gamma(-1/2)}{\pi^{3/2}} = -\frac{2}{\pi} \quad \text{is transcendental}$$

▷ If $a_0 = 0, a_1 = 1, (2n+1)a_{n+2} - (7n+11)a_{n+1} + (2n+1)a_n = 0$, then $f = \sum_n a_n t^n$ is transcendental, since $a_n \sim C\left(\frac{7+\sqrt{33}}{4}\right)^n n^{\sqrt{75/44}}$ with $C \approx 0.56$.

More Criteria:
rational-transcendental dichotomies

Classifications of algebraic series

- Algebraic series with algebraic Hadamard inverse are rational
- Algebraic series with bounded integer coefficients are rational
- Algebraic series that satisfy Mahler equations are rational
- Algebraic series with coefficients in a finite set are rational
- Algebraic series with multiplicative coefficients are rational

Resulting transcendence criteria

Let $f \in \mathbb{Q}[[t]]$ be an irrational power series.

- If f has algebraic Hadamard inverse
- If f has bounded integer coefficients
- If f satisfies a Mahler equation
- If f has coefficients in a finite set
- If f has multiplicative coefficients

then f is transcendental

Algebraic series that satisfy Mahler equations

Theorem [Nishioka, 1996]
Assume $f \in \mathbb{Q}[[t]]$ satisfies a k-Mahler equation ($k \geq 2$)

$$c_0(t)f(t) + c_1(t)f(t^k) + \cdots + c_r(t)f(t^{k^r}) = 0,$$

where $c_i(t) \in \mathbb{Q}[t]$ with $c_0 c_r \neq 0$. Then f is either transcendental, or rational.

▷ Thue-Morse series T is transcendental: $T(t) = (1-t)T(t^2)$

▷ Baum–Sweet series B is transcendental: $B(t) = tB(t^2) + B(t^4)$

▷ Rudin-Shapiro series R is transcendental: $R(t) = (1-t)R(t^2) + 2tR(t^4)$

▷ Stern series S is transcendental: $tS(t) = (t^2+t+1)S(t^2)$

▷ Bonus: "transcendental" can be replaced by "non-D-finite" [Bézivin, 1994], and even by admits the unit circle as a natural boundary [Randé, 1992]

▷ Bonus: algorithms for existence of rational solutions [Bell, Coons, 2015]

Power series with (polynomially) bounded integer coefficients

Theorem [Borel, 1894], [Fatou, 1904]
Any $f = \sum_n f_n t^n \in \mathbb{Z}[[t]]$ with $f_n = O(n^d)$ for some $d \geq 0$ is

1. either transcendental;
2. or rational, of the form $P(t)/(1-t^m)^n$, with $P(t) \in \mathbb{Z}[t]$ and $m, n \in \mathbb{N}$.

Alternative form: Convergence radius of algebraic series in $\mathbb{Z}[[t]]$ is ≤ 1; equality holds only for rational functions whose poles are all roots of unity.

▷ The constant 1 is the best possible: for any $r < 1$, there exists a power series $A_r \in \mathbb{Z}[[t]]$ algebraic irrational with convergence radius r.

$$\frac{1}{\sqrt{1-4t^{\ell}}} = \sum_n \binom{2n}{n} t^{\ell n}$$

▷ Bonus [Pólya, 1916], [Carlson, 1916]: one can replace "is transcendental" in the conclusion by "is non-D-finite", and even by "admits the unit circle as a natural boundary" (i.e. "has no analytic continuation beyond the unit disc").

Power series with coefficients in a finite set

Theorem [Carlson, 1918], [Szegö, 1922]
A power series $f \in \mathbb{Q}[[t]]$ with only finitely many distinct coefficients is:

1. either transcendental;
2. or rational, of the form $P(t)/(1-t^m)$, with $P(t) \in \mathbb{Q}[t]$ and $m \in \mathbb{N}$.

▷ One can replace "is transcendental" in the conclusion of the Carlson-Szegö theorem by "is non-D-finite", and even by "admits the unit circle as a natural boundary" (i.e. "has no analytic continuation beyond the unit disc").

▷ Recent extension [Bell, Chen, 2016]: a multivariate D-finite power series with coefficients from a finite set is rational.

Power series with multiplicative coefficients

Theorem [Sárközy, 1978], [Bézivin, 1995], [Bell, Bruin, Coons, 2012]
A series with multiplicative coefficients is either transcendental or rational: Let $a : \mathbb{N} \to \mathbb{Q}$ satisfy $a(mn) = a(m)a(n)$ for all coprime $m, n \in \mathbb{N} \setminus \{0\}$. If $F = \sum_n a(n)t^n$ is algebraic, then either f is a polynomial, or there exist $k \in \mathbb{N}$ and a periodic multiplicative $\chi : \mathbb{N} \to \mathbb{Q}$ such that $a(n) = n^k \chi(n)$.

▷ Holds over any field of characteristic zero.

▷ Holds if "transcendental" is replaced by "non-D-finite".

▷ Via [Banks, Luca, Shparlinski, 2005], proves transcendence of $\sum a(n)t^n$ for: $a \in \{\varphi \,(\text{Euler totient}), \mu \,(\text{Möbius}), \lambda \,(\text{Liouville}), \sigma_k \,(\text{divisors power sum})\}$.

▷ Ramanujan's modular discriminant is transcendental.

Research problem: characterize all D-finite power series whose coefficient sequence has the p-Lucas property for "many" primes p.

▷ For "generic" $a, b, c \in \mathbb{Z}$, the Apery-like differential equations (in $\theta = t\frac{d}{dt}$)

$$\theta^2 - t(a\theta^2 + a\theta + b) + ct^2(\theta+1)^2, \qquad \theta^3 - t(2\theta+1)(\hat{a}^2\theta^2 + \hat{a}\theta + \hat{b}) + \hat{c}t^2(\theta+1)^3$$

do not admit any power series solution $\sum_n A_n t^n$ with integer coefficients.

▷ Exceptions: sporadic binomial sums [Beukers, Zagier; Almkvist, Zudilin]

(a) $a = 7,\ b = 2,\ c = -8,\quad A_n = \sum_k \binom{n}{k}^3;$

(b) $a = 11, b = 3,\ c = -1,\quad A_n = \sum_k \binom{n}{k}^2\binom{n+k}{n};$

(c) $a = 10, b = 3,\ c = 9,\quad A_n = \sum_k \binom{n}{k}^2\binom{2k}{k};$

(d) $a = 12, b = 4,\ c = 32,\quad A_n = \sum_k \binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k};$

(f) $a = 9,\ b = 3,\ c = 27,\quad A_n = \sum_k (-1)^k 3^{n-3k}\binom{n}{3k}\frac{(3k)!}{k!^3};$

(g) $a = 17, b = 6,\ c = 72,\quad A_n = \sum_{k,l} (-1)^k 8^{n-k}\binom{n}{k}\binom{k}{l}^3.$

(δ) $\hat{a} = 7,\ \hat{b} = 3,\ \hat{c} = 81,\quad A_n = \sum_k (-1)^k 3^{n-3k}\binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3};$

(η) $\hat{a} = 11, \hat{b} = 5,\ \hat{c} = 125,\quad A_n = \sum_k (-1)^k\binom{n}{k}^3\left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n}\right);$

(α) $\hat{a} = 10, \hat{b} = 4,\ \hat{c} = 64,\quad A_n = \sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k};$

(ϵ) $\hat{a} = 12, \hat{b} = 4,\ \hat{c} = 16,\quad A_n = \sum_k \binom{n}{k}^2\binom{2k}{n}^2;$

(ζ) $\hat{a} = 9,\ \hat{b} = 3,\ \hat{c} = -27,\quad A_n = \sum_{k,l} \binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n};$

(γ) $\hat{a} = 17, \hat{b} = 5,\ \hat{c} = 1,\quad A_n = \sum_k \binom{n}{k}^2\binom{n+k}{n}^2.$

Theorem [Malik, Straub, 2015] All these 12 sequences are p-Lucas.

"Certain differential equations look better than others, at least arithmetically"

Bonus: Constant-term sequences

Theorem [Samol, van Straten, 2009] Let $\Lambda(x_1, \ldots, x_d) \in \mathbb{Q}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ such that the Newton polyhedron of Λ has the origin as its only interior integral point. Then the constant-term sequence

$$a_n = \left[x_1^0 \cdots x_d^0\right] \Lambda^n$$

is p-Lucas for any prime p.

- $\sum_n \binom{n}{k}^2 \binom{n+k}{k}$ for $\Lambda = \frac{(y+1)(x+1)(x+y+1)}{xy} = 3+x+y+2\left(\frac{1}{x}+\frac{1}{y}\right)+\frac{x}{y}+\frac{y}{x}+\frac{1}{xy}$

- $\sum_n \binom{n}{k}^2 \binom{n+k}{k}^2$ for $\Lambda = \frac{(x+y)(z+1)(x+y+z)(y+z+1)}{xy}$

Bonus: Diagonal sequences

Theorem [Rowland, Yassawi, 2015] If $P(x_1,\ldots,x_d) \in \mathbb{Q}[x_1,\ldots,x_d]$ has degree at most 1 in each x_i, and $P(0,\ldots,0)=0$, then the diagonal sequence

$$a_n = [x_1^n \cdots x_d^n]\ \frac{1}{P}$$

is p-Lucas for any prime p.

- $\sum_n \binom{n}{k}^2 \binom{n+k}{k}$ is the diagonal sequence of $\dfrac{1}{(1-x-y)(1-z)-xyz}$
- $\sum_n \binom{n}{k}^2 \binom{n+k}{k}^2$ is the diagonal sequence of $\dfrac{1}{(1-x-y)(1-z-t)-xyzt}$
- $\sum_n \binom{n}{k}^d$ is the diag. seq. of $\dfrac{1}{(1-x_1)(1-x_2)\cdots(1-x_d)-x_1x_2\cdots x_d}$
- $\sum_n \binom{n}{k}^2 \binom{2k}{n}^2$ is the diagonal sequence of $\dfrac{1}{1-e_1+2e_3+4e_4}$,

where e_j is the j-th elementary symmetric function in x_1,x_2,x_3,x_4

- $\sum_n \binom{n}{k}^2 \binom{n+k}{k}^3$ is the diag. seq. of $\dfrac{1}{1-(xyz+xy+xz+yz+z)(uv+u+v)}$
- $\sum_n \binom{n}{k}^3 \binom{n+k}{k}^2$ diag. seq. of $\dfrac{1}{1-(xyz+xy+xz+yz+y+z)(uv+u+v)}$

Thanks for your attention!